nent place; yet, according to an account by Lampe*, Jacob Steiner advised his students especially to study the theory of numbers on the ground that this subject was eminently suited to cultivate acuteness. Steiner is said to have reprimanded severely a young student who wanted to devote himself exclusively to the study of synthetic geometry as Steiner had done himself. It is said that Steiner remarked that not all those who would say to him, Lord, Lord, could enter the Kingdom of Heaven.

The scientific developments of the fundamental facts of the theory of numbers are largely due to Gauss and they were first published in the classic Disquisitiones Arithmeticae, 1801. In a biographical sketch of Gauss entitled *Gauss zum Gedaechtniss*, 1856, Waltershausen stated that Gauss called mathematics the queen of the sciences and arithmetic the queen of mathematics. Although this science descends often to render service to astronomy and other natural sciences, yet it deserves under all conditions the first rank.

"Goepel was first attracted by the higher theory of numbers, like many of those who are chosen for mathematical speculations."† "There is a theory which has been equally useful to me in all my researches, namely, that of the groups formed by the linear substitutions. In fact, these substitutions play a preponderant role in the study of linear equations and in that of arithmetic forms. It is to this circumstance that one ought to attribute the interrelations, often unexpected, which I shall note later between the theory of numbers and the theory of Fuchsian functions,—theories which, moreover, do not at first appear to have any point of contact."‡ "From the problems which we have just examined we see that the three fundamental branches of mathematics, namely, the theory of numbers, algebra, and the theory of functions, are most intimately related; and I am convinced that the theory of analytic functions of several variables will make decided progress if one shall arrive at the discovery and the study of functions which, in the domain of any given algebraic numbers, play a role analogous to that played by the expotential functions in the domain of the rational numbers, and by the elliptic modular functions in the domain of quadratic imaginary numbers."§

It would be easy to extend this list of quotations very much but the object of the present note is attained if a sufficient number of quotations have been given to exhibit the importance and the bearing of the theory of numbers. Just as the engineer is inclined to hasten into his profession without a sufficient training in mathematics, so the student of mathematics is often tempted to proceed to fields for which he has been imperfectly prepared. While many mathematical subjects are mutually helpful, the question as to which should be regarded as the more fundamental is sometimes of great importance.

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* Lampe, *Bibliotheca Mathematica*, 3rd Series, vol. 1 (1900), p. 134.

† C. G. J. Jacobi, *Crelle Journal*, vol. 35 (1847), p. 313.

‡ Poincaré, *Note sur les travaux scientifiques* (1884), p. 7.

§ Hilbert, *Paris Intenational Mathematical Congress* (1900), p. 90.

A GEOMETRIC EXAMPLE OF AN INDETERMINATE FORM.

By ARTHUR C. LUNN, University of Chicago.

In the prolate ellipsoid of revolution generated by turning the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ around the x-axis be cut by the plane $x=c$ through one focus, the area of the smaller part, between this focal plane and the nearer vertex, is given by the integral

$$S=\frac{2\pi b}{a^2}\int_c^a \sqrt{[a^4-(a^2-b^2)x^2]}\,dx.$$

When the constants are expressed in terms of the eccentricity e and the distance q between the focus and vertex, by means of the relations

$$c=ae,\ b=a\sqrt{[1-e^2]},\ q=a(1-e),$$

the integral reduces to

$$(1)\qquad S=\pi q^2\sqrt{\frac{1+e}{(1-e)^3}}\{\sqrt{[1-e^2]}-e\sqrt{[1-e^4]}+\frac{\sin^{-1}e-\sin^{-1}e^2}{e}\}.$$

If now the distance q be kept constant, the limiting surface (for $e=1$) is the paraboloid of revolution generated by $y^2=4qx$ (with a convenient change of axes); and the corresponding area is

$$S_1=\frac{\pi}{q}\int_0^{2q} y\sqrt{[y^2+4q^2]}\,dy$$

which reduces to

$$(2)\qquad S_1=\frac{8\pi q^2}{3}(\sqrt{8}-1).$$

It is thus apparent that as e approaches unity the limit S should be S_1. But the expression for S is then formally indeterminate, and verification is not directly practicable by the customary method of successive differentiation with respect to e, the reason being the presence of the factor $(1-e)$ raised to a fractional power, which cannot be reduced to a constant by repeated differentiation. It may be noticed also that the point $e=1$ is a branch-point for $\sin^{-1}e$ and for $\sin^{-1}e^2$, where two values of a many-valued function come together.

This suggests the change of variable $1-e=t^2$; then S as a function of t may be treated by successive differentiation; or, what is at bottom the same thing, let the various terms in the brace be expanded in integral powers of t, with the values at $t=0$ suitably chosen thus:

$$\sqrt{[1-e^2]}=\sqrt{[2]}t-\tfrac{1}{12}\sqrt{[2]}t^3....$$

$$e\sqrt{[1-e^4]}=2t-\tfrac{1}{2}t^3....$$

$$\sin^{-1}e=\frac{\pi}{2}-\sqrt{[2]}t-\frac{\sqrt{2}}{12}t^3....$$

$$\sin^{-1}e^2=\frac{\pi}{2}-2t+\tfrac{1}{6}t^3....$$

then in the expression for S, the factors t^3, to which the indeterminateness is due, may be cancelled, and the leading coefficients are then such as to show that the limit of S is S_1 as t approaches zero.

A contrasting case occurs with respect to the entire area of the ellipsoid, which is

$$2\pi a^2\sqrt{[1-e^2]}\{\frac{\sin^{-1}e}{e}+\sqrt{[1-e^2]}\},$$

the limit of which, as e approaches zero, is the area of a sphere of radius a. Here the indeterminateness occurring in $\frac{\sin^{-1}e}{e}$ is of integral order.

Many similar examples could be cited, for instance, that of the center of gravity or center of pressure of part of an ellipse and the corresponding part of the limiting parabola, where the limit of an indeterminate form, discovered by geometric or physical considerations, is to be analytically verified.

CERTAIN INTEGRATION FORMULAE USEFUL IN NUMERICAL COMPUTATION.

By S. A. COREY, Hiteman, Iowa.

In developing a function of a variable by Taylor's, Maclaurin's, or the Euler-Maclaurin formulae, the evaluation of the successive derivatives is a fundamental and in most cases indispensable operation. In many instances the higher derivatives become exceedingly complex and their evaluation correspondingly difficult. In such cases frequently the labor involved in finding results of the required degree of accuracy by these formulae becomes prohibitively great, and other methods are resorted to, involving usually some of the formulae of the Calculus of Finite Differences. The formulae given below, some of which are probably new and others old except as to the remainder term, will, it is believed, prove useful in many such cases, for the reason that the use of the higher derivatives is not involved except in the remainder term, in which term a roughly approximate value usually suffices. In certain cases it will be found desirable to develop the given function to a limited number of terms by Taylor's or Maclaurin's formula, to differentiate this series the number of times required to get an expression for the derivative involved in the remainder term, and then from this expression to determine roughly the value of the remainder term. Another method of indicating the degree of accuracy assured in cases where the value of the remainder term is not easily determined is to develop the function by two of the formulae given below, and then to compare the results obtained. It is usually safe to assume that the difference in the results obtained may be substituted for the remainder term in either of the two formulae employed. It will be noted that the remainder term, as given below, is expressed in a form similar to that frequently employed in Taylor's formula, and that it is therefore always possible to compare the accuracy assured with the accuracy assured by the use of Taylor's formula.

From the manner of deriving these formulae it will appear that it is not practicable to give all the formulae of like form, obtainable from the general formula (c), but the list here given is, probably, sufficiently complete to satisfy all ordinary needs. Should others be desired they may, of course, be obtained from the general formula. In fact, were it not for the considerable amount of labor involved in deriving these formulae one at a time, all these formulae could be thus obtained and there would be no occasion for appending the following rather formidable list.

These formulae are all based on the so-called Euler-Maclaurin formula, a semi-convergent formula which has been recently studied by E. Borel, E. W. Barnes,* and other European mathematicians, but to which but scanty

* Borel, *Lecons sur les sèries divergentes*, Chap. III.
Borel, *Acta Mathematica*, Vol. XX, page 360.
Barnes, *Quarterly Journal of Mathematics*, Vol. XXXV, pages 175-188.
Barnes, *Proc. London Mathematical Society*, Series 2, Vol. 3, page 253.
Cantor, *Geschichte der Mathematik*, Vol. 3, pages 635, 661.

reference is made in our American mathematical literature. It would seem a greater use of this formula in our American text books would be advisable.

William Fleetfoot Sheppard, in his article on "Mensuration" in the 11th edition of the Encyclopedia Brittannica, volume XVIII, pages 142-3-4, paragraphs 68 to 79 inclusive, gives a very satisfactory reference to this class of formulae, to which the reader is referred. In paragraph 71 he gives a general method of constructing some of these formulae, but it should be carefully noted that the subscripts of C which he uses are determined by a system of notation somewhat different from that herein used. In paragraph 79 he gives another general form involving an expression for the remainder, but the use of the formulae in general numerical work is not emphasized, and the reader is likely to infer that their use is confined chiefly to the subject under discussion.

The Euler-Maclaurin formula may be written,

$$f(a+x)=f(a)+\frac{x}{2m}\{f'(a)+f'(a+x)+2[f'(a+\frac{x}{m})+f'(a+\frac{2x}{m})$$

$$+f'(a+\frac{3x}{m})....+f'(a+\frac{m-1}{m}x)]\}-\frac{B_1 x^2}{m^2.2!}[f''(a+x)-f''(a)]$$

$$+\frac{B_2 x^4}{m^4.4!}[f^{iv}(a+x)-f^{iv}(a)]....$$

$$+(-1)^n\frac{B_n x^{2n}}{m^{2n}.(2n)!}[f^{2n}(a+x)-f^{2n}(a)]+S....(a).$$

B_n being Bernoulli's nth number.

Expressed briefly and for clearness in what follows, this becomes:

$$f(a+x)=F(x, m)+\frac{k_2}{m^2}+\frac{k_4}{m^4}+\frac{k_6}{m^6}+\frac{k_8}{m^8}....+\frac{k_{2n}}{m^{2n}}+S....(b),$$

in which k_2, k_4, k_6, ..., k_{2n}, are independent of m, and S is an expression for the remainder after the general term.

If $(r+1)$ values of m be taken, r of the quantities k, say k_{2i}, $k_{2(i+1)}$, $k_{2(i+2)}$, ..., $k_{2(i+r-1)}$, may be eliminated between the $(r+1)$ resulting equations, and the result expressed in determinant form as follows:

$$f(a+x)=\frac{\begin{vmatrix} M_1+S_1 & m_1^{-2i} & m_1^{-2(i+1)} & m_1^{-2(i+2)} & \cdots & m_1^{-2(i+r-1)} \\ M_2+S_2 & m_2^{-2i} & m_2^{-2(i+1)} & m_2^{-2(i+2)} & \cdots & m_2^{-2(i+r-1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ M_{(r+1)}+S_{(r+1)} & m_{(r+1)}^{-2i} & m_{(r+1)}^{-2(i+1)} & m_{(r+1)}^{-2(i+2)} & \cdots & m_{(r+1)}^{-2(i+r-1)} \end{vmatrix}}{\begin{vmatrix} 1 & m_1^{-2i} & m_1^{-2(i+1)} & m_1^{-2(i+2)} & \cdots & m_1^{-2(i+r-1)} \\ 1 & m_2^{-2i} & m_2^{-2(i+1)} & m_2^{-2(i+2)} & \cdots & m_2^{-2(i+r-1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & m_{(r+1)}^{-2i} & m_{(r+1)}^{-2(i+1)} & m_{(r+1)}^{-2(i+2)} & \cdots & m_{(r+1)}^{-2(i+r-1)} \end{vmatrix}} \quad \ldots\ldots(c),$$

where M_1, M_2, ..., $M_{(r+1)}$, represent the approximate values of $f(a+x)$, obtained by (b) for m_1, m_2, ..., $m_{(r+1)}$, respectively, by employing all the terms preceding the term $\frac{k_{2i}}{m^{2i}}$, and where S_1, S_2, ..., $S_{(r+1)}$, represent the corresponding remainder terms, *i. e.* the remainder after the term $\frac{k_{2(i+r-1)}}{m^{2(i+r-1)}}$ in each of the $(r+1)$ equations.

Splitting up the determinant of the numerator into two other similar determinants, one containing the M elements and the other the S elements, we have, by letting D represent the determinant,

$$f(a+x)=D_M/D+D_S/D.... \text{(d)}.$$

To rid the indices of the elements of these determinants of the minus sign, divide each element in a row by the last element in that row. Now reduce the order by taking the co-factors of all the elements in the first column. These co-factors have evidently as factors $(m^2{}_u-m^2{}_v)$, u and v taking any value assumed by the subscripts of m, except that of the element, the co-factor of which is involved, and except $u=v$. Their evaluation follows readily.

The remainders, S, used in determining the remainders in the following formulae, were obtained from the solution of problem 237, Calculus, given in the June-July, 1907, number of THE AMERICAN MATHEMATICAL MONTHLY, to which reference must be had for an understanding of the method used in their derivation. Each remainder term in the following is dependent for its value on but *one* higher derivative, and for that reason the limits of its value are fixed, while the form given by Mr. Sheppard is a function of *all* derivatives higher than a certain derivative, and usually becomes *divergent* if a sufficiently large number of terms be taken.

Instead of retaining in the following formulae the M_1, M_2, ..., $M_{(r+1)}$, used in (c), it is convenient to employ a slightly different notation, viz., C_m represents the chordal area of the curve $y=f(a+x)$, where the

value of y is taken at each of the $(m+1)$ equidistant intervals, 0, x/m, $2x/m$, $3x/m$, ..., $(m-1)x/m$ and x, thus

$$C_m=\frac{x}{m}[\frac{f'(a)+f'(a+x)}{2}+f'(a+\frac{x}{m})+f'(a+\frac{2x}{m})+....+f'(a+\frac{(m-1)}{m}x)].$$

In the formulae involving y, the first extreme ordinate is invariably y_o, and the last extreme ordinate y_n, where n has the highest value of the subscript given in that formula. The distances of the intermediate ordinates y_r (where $o<r<n$) and the last ordinate y_n from the first extreme ordinate are *proportional to their subscripts.* The ordinates are therefore not *all* equidistant *in cases where the smaller values of m are not all factors of the largest value.*

Where C is given more than one subscript, thus $C_{m,t}$, it is understood that to the expression for the chordal area C_m has been added the sum of the first t terms of the series $\frac{k_2}{m^2}+\frac{k_4}{m^4}+\frac{k_6}{m^6}+$ etc., in (b), where

$$k_{2n}=+(-1)^n\frac{B_nx^{2n}}{(2n)!}[f^{2n}(a+x)-f^{2n}(a)],$$

B_n being Bernoulli's nth number.

$$A\equiv\int_0^x f(a+x)\,dx\equiv\int_0^x y dx,\text{ and }R_{2n}\equiv\frac{x^{2n}}{(2n)!}[f^{2n}(a+\theta_1x)-f^{2n}(a+\theta_2x)],$$

where both θ_1 and θ_2 have some value greater then zero and less than unity.

$A=\frac{1}{8}(4C_2-C_1)\pm\frac{4}{15}R_4$ . . . . . (1).

$A=\frac{1}{8}(9C_3-C_1)\pm\frac{4}{45}R_4$ . . . . . (2).

$A=\frac{1}{45}(64C_4-20C_2+C_1)\pm\frac{3}{70}R_6$ . . . . . (3).

$A=\frac{1}{120}(243C_3-128C_2+5C_1)\pm\frac{2}{21}R_6$ . . . . . (4).

$A=\frac{1}{2520}(8192C_4-6561C_3+896C_2-7C_1)\pm.0229R_8$ . . . . (5).

$A=\frac{1}{360}(512C_{12}-175C_6+14C_3-C_2)\pm.000,14R_8$ . . . . (6).

$A=\frac{1}{35}(56C_{12}-28C_6+8C_4-C_3)\pm.000,099R_8$ . . . . . (7).

$A=\frac{1}{10}(15C_6-6C_3+C_2)\pm.006,62R_6$ . . . . . (8).

$A=\frac{1}{2520}(567C_6-512C_4+162C_3-7C_2)\pm.002,55R_8$ . . . . (9).

$A=\frac{1}{42005}(10,368C_6-8,192C_4+2,025C_3-C_1)\pm.001,93R_8$ . . . (10).

$$A=\tfrac{1}{3150}(6,561C_6-3,584C_4+175C_2-2C_1)\pm.003,77R_8 \quad . \quad . \quad . \quad (11).$$

$$A=\tfrac{1}{840}(1,296C_6-567C_3+112C_2-C_1)\pm.008,39R_8 \quad . \quad . \quad . \quad . \quad (12).$$

$$A=\tfrac{1}{126000}(34,992C_6-32,768C_4+10,935C_3-560C_2+C_1)\pm.002,74R_{10} \quad (13).$$

$$A=\tfrac{1}{2835}(4,096C_8-1,344C_4+84C_2-C_1)\pm.002,02R_8 \quad . \quad . \quad . \quad (14).$$

$$A=\tfrac{1}{87381800}(33,554,432C_8-31,177,872C_6+7,569,408C_4-1,240,029C_3$$
$$+25,872C_2-11C_1)\pm.000,225R_{12} \quad . \quad . \quad . \quad . \quad . \quad . \quad (15).$$

$$A=\tfrac{1}{1801800}(2,985,984C_{12}-1,667,952C_6+585,728C_4-104,247C_3$$
$$+2,288C_2-C_1)\pm.000,095,6R_{12} \quad . \quad . \quad . \quad . \quad . \quad . \quad (16).$$

$$A=\tfrac{1}{10150}(1,728C_{12}-945C_6+320C_4-54C_3+C_2)\pm.000,071,2R_{10} \quad . \quad (17).$$

$$A=\tfrac{1}{280}(384C_6-105C_3+C_1)\pm.006,05R_6 \quad . \quad . \quad . \quad . \quad . \quad . \quad (18).$$

$$A=C_6+\tfrac{1}{35}(C_6-C_3)-\tfrac{1}{280}(C_3-C_1)\pm.006,05R_6 \quad . \quad . \quad . \quad . \quad (18a).$$

$$A=\tfrac{1}{210}(243C_6-35C_2+2C_1)\pm.017,1R_6 \quad . \quad . \quad . \quad . \quad . \quad . \quad (19).$$

$$A=C_6+\tfrac{1}{70}(C_6-C_2)-\tfrac{1}{105}(C_2-C_1)\pm.017,1R_6 \quad . \quad . \quad . \quad . \quad (19a).$$

$$A=\tfrac{1}{45}(64C_8-20C_4+C_2)\pm.001,34R_6 \quad . \quad . \quad . \quad . \quad . \quad . \quad (20).$$

$$A=\tfrac{1}{72291925}(1,048,576C_{16}-348,160C_8+22,848C_4-340C_2+C_1)$$
$$\pm.000,030,2R_{10} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21).$$

$$A=\tfrac{1}{65270205000}(110,075,314,176C_{24}-64,900,362,240C_{12}+25,081,937,920C_8$$
$$-5,178,990,960C_6+202,076,160C_4-9,808,695C_3+38,640C_2-C_1)$$
$$\pm.000,000,134R_{16} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22).$$

$$A=\tfrac{1}{23760}(3,200C_{10}-825C_5+C_1)\pm.000,624R_6 \quad . \quad . \quad . \quad . \quad . \quad (23).$$

$$A=\tfrac{1}{2010993768}(3,648,810,176C_{18}-1,674,098,987C_9+36,837,504C_6$$
$$-575,586C_3+20,672C_2-11C_1)\pm.000,000,736R_{12} \quad . \quad . \quad . \quad (24).$$

Formulae involving dy/dx.

$$A=\tfrac{1}{15}(16C_{2,1}-C_{1,1})\pm.115R_6 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25).$$

$$A=C_{2,1}+\tfrac{1}{15}(C_{2,1}-C_{1,1})\pm.115R_6 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25a).$$

$$A=\tfrac{1}{80}(81C_{3,1}-C_{1,1})\pm.019,1R_6 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26).$$

$$A=C_{3,1}+\tfrac{1}{80}(C_{3,1}-C_{1,1})\pm.019,1R_6 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26a).$$

$$A=\tfrac{1}{945}(1,024C_{4,1}-80C_{2,1}+C_{1,1})\pm.005,9R_8 \quad . \quad . \quad . \quad . \quad . \quad . \quad (27).$$

$$A=C_{4,1}+\tfrac{79}{945}(C_{4,1}-C_{2,1})-\tfrac{1}{945}(C_{2,1}-C_{1,1})\pm.005,9R_8 \quad . \quad . \quad . \quad . \quad (27a).$$

$$A=\tfrac{1}{1680}(2,187C_{3,1}-512C_{2,1}+5C_{1,1})\pm.019,6R_8 \quad . \quad . \quad . \quad . \quad . \quad (28).$$

$$A=\tfrac{1}{75600}(131,072C_{4,1}-59,049C_{3,1}+3,584C_{2,1}-7C_{1,1})\pm.002,88R_{10} \quad . \quad (29).$$

$$A=\tfrac{1}{33775}(36,864C_{12,1}-3,150C_{6,1}+63C_{3,1}-2C_{2,1})\pm.000,002,76R_{10} \quad . \quad (30).$$

$$A=\tfrac{1}{245}(270C_{6,1}-27C_{3,1}+2C_{2,1})\pm.000,389R_8 \quad (31).$$

$$A=\tfrac{1}{8825}(10,206C_{6,1}-4,096C_{4,1}+729C_{3,1}-14C_{2,1})\pm.000,151R_{10} \quad (32).$$

$$A=\tfrac{1}{280400}(373,248C_{6,1}-131,072C_{4,1}+18,225C_{3,1}-C_{1,1})\pm.000,118R_{10} \quad (33).$$

$$A=\tfrac{1}{89775}(118,098C_{6,1}-28,672C_{4,1}+350C_{2,1}-C_{1,1})\pm.000,25R_{10} \quad (34).$$

$$A=\tfrac{1}{42000}(46,656C_{6,1}-5,103C_{3,1}+448C_{2,1}-C_{1,1})\pm.000,633R_{10} \quad (35).$$

$$A=\tfrac{1}{831600}(1,259,712C_{6,1}-524,288C_{4,1}+98,415C_{3,1}-2,240C_{2,1}+C_{1,1})$$
$$\pm.000,202R_{12} \quad (36).$$

$$A=\tfrac{1}{240975}(262,144C_{8,1}-21,504C_{4,1}+336C_{2,1}-C_{1,1})\pm.000,089,8R_{10} \quad (37).$$

$$A=\tfrac{1}{11351134000}(2,147,483,648C_{8,1}-1,122,403,392C_{6,1}+121,110,528C_{4,1}$$
$$-11,160,261C_{3,1}+103,488C_{2,1}-11C_{1,1})\pm.000,010,7R_{14} \quad (38).$$

$$A=\tfrac{1}{37833778000}(429,981,696C_{12,1}-60,046,272C_{6,1}+9,371,648C_{4,1}$$
$$-938,223C_{3,1}+9,152C_{2,1}-C_{1,1})\pm.000,002,8R_{14} \quad (39).$$

$$A=\tfrac{1}{1009725}(124,416C_{12,1}-17,010C_{6,1}+2,560C_{4,1}-243C_{3,1}+2C_{4,1})$$
$$\pm.000,001,66R_{12} \quad (40).$$

$$A=\tfrac{1}{2991376}(320,000C_{10,1}-20,625C_{5,1}+C_{1,1})\pm.000,014,3R_8 \quad (41).$$

Formulae involving $\frac{dy}{dx}$, $\frac{d^3y}{dx^3}$ and $\frac{d^5y}{dx^5}$.

$$A=\tfrac{1}{63}(64C_{2,2}-C_{1,2})\pm.078,4R_8 \quad (42).$$

$$A=C_{2,2}+\tfrac{1}{63}(C_{2,2}-C_{1,2})\pm.078,4R_8 \quad (42a).$$

$$A=\tfrac{1}{728}(729C_{3,2}-C_{1,2})\pm.006,03R_8 \quad (43).$$

$$A=C_{3,2}+\tfrac{1}{728}(C_{3,2}-C_{1,2})\pm.006,03R_8 \quad (43a).$$

$$A=\tfrac{1}{18085}(16384C_{4,2}-320C_{2,2}+C_{1,2})\pm.001,31R_{10} \quad (44).$$

$$A=\tfrac{1}{17640}(19,683C_{3,2}-2,048C_{2,2}+5C_{1,2})\pm.007,05R_{10} \quad (45).$$

$$A=C_{3,2}+\tfrac{2227}{19660}(C_{3,2}-C_{2,2})-\tfrac{1}{35328}(C_{2,2}-C_{1,2})\pm.007,05R_{10} \quad (45a).$$

$$A=\tfrac{1}{1580640}(2,097,152C_{4,2}-531,441C_{3,2}+14,336C_{2,2}-7C_{1,2})\pm.000,669R_{12} \quad (46).$$

$$A=\tfrac{1}{5191555}(5,308,416C_{12,2}-113,400C_{6,2}+567C_{3,2}-8C_{2,2})$$
$$\pm.000,000,086,3R_{12} \quad (47).$$

$$A=\tfrac{1}{94885}(9,720C_{6,2}-243C_{3,2}+8C_{2,2})\pm.000,037,9R_{10} \quad (48).$$

$$A=C_{6,2}+\tfrac{147}{18907}(C_{6,2}-C_{3,2})-\tfrac{8}{94885}(C_{3,2}-C_{2,2})\pm.000,037,9R_{10} \quad (48a).$$

$$A=\tfrac{1}{37833778000}(45,349,632C_{6,2}-8,388,608C_{4,2}+885,735C_{3,2}-8,960C_{2,2}$$
$$+C_{1,2})\pm.000,027,3R_{14} \quad (49).$$

$$A=\tfrac{1}{1635480}(1,679,616C_{6,2}-45,927C_{3,2}+1,792C_{2,2}-C_{1,2})\pm.000,079R_{12} \quad (50).$$

$$A=\tfrac{1}{1643445}(16,777,216C_{8,2}-344,064C_{4,2}+1,344C_{2,2}-C_{1,2})$$
$$\pm.000,006,39R_{12} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51).$$

$$A=\tfrac{1}{59897237400}(61,917,364,224C_{12,2}-2,161,665,792C_{6,2}+149,946,368C_{4,2}$$
$$-8,444,007C_{3,2}+36,608C_{2,2}-C_{1,2})\pm.000,000,108R_{16} \quad . \quad . \quad (52).$$

$$A=\tfrac{1}{255}(256C_{2,3}-C_{1,3})\pm.073,1R_{10} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53).$$

$$A=C_{2,3}+\tfrac{1}{255}(C_{2,3}-C_{1,3})\pm.073,1R_{10} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53a).$$

$$A=\tfrac{1}{6560}(6,561C_{3,3}-C_{1,3})\pm.002,53R_{10} \quad . \quad . \quad . \quad . \quad . \quad . \quad (54).$$

$$A=C_{3,3}+\tfrac{1}{6560}(C_{3,3}-C_{1,3})\pm.002,53R_{10} \quad . \quad . \quad . \quad . \quad . \quad . \quad (54a).$$

$$A=\tfrac{1}{260865}(262,144C_{4,3}-1,280C_{2,3}+C_{1,3})\pm.000,391R_{12} \quad . \quad . \quad . \quad (55).$$

$$A=C_{4,3}+\tfrac{1}{260865}[1,279(C_{4,3}-C_{2,3})-(C_{2,3}-C_{1,3})]\pm.000,391R_{12} \quad . \quad . \quad (55a).$$

$$A=\tfrac{1}{60066000}(60,466,176C_{6,3}-413,343C_{3,3}+7,168C_{2,3}-C_{1,3})$$
$$\pm.000,013,3R_{14} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56).$$

Formulae expressed in terms of ordinates.

$$A=\frac{x}{6}[(y_0+y_2)+4y_1]\pm\tfrac{4}{15}R_4 \text{ (Simpson's first formula)} \quad . \quad . \quad (1r).$$

$$A=\frac{x}{8}[(y_0+y_3)+3(y_1+y_2)]\pm\tfrac{4}{45}R_4 \text{ (Simpson's second formula)} \quad . \quad (2r).$$

$$A=\frac{x}{90}[7(y_0+y_4)+32(y_1+y_3)+12y_2]\pm\tfrac{8}{945}R_6 \quad . \quad . \quad . \quad . \quad . \quad (3r).$$

$$A=\frac{x}{120}[11(y_0+y_6)+81(y_2+y_4)-64y_3]\pm\tfrac{8}{2\,1\,1}R_6 \quad . \quad . \quad . \quad . \quad (4r).^*$$

$$A=\frac{x}{2520}[151(y_0+y_{12})+2,048(y_3+y_9)+2,496y_6-2,187(y_4+y_8)$$
$$\pm.022,9R_8 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5r).^*$$

$$A=\frac{x}{2100}[53(y_0+y_{12})+256(y_1+y_3+y_5+y_7+y_9+y_{11})+81(y_2+y_{10})$$
$$+78y_6+109(y_4+y_8)]\pm.000,14R_8 \quad . \quad . \quad . \quad . \quad . \quad . \quad (6r).$$

$$A=\frac{x}{210}[5(y_0+y_{12})+28(y_1+y_5+y_7+y_{11})+40(y_3+y_9)+12y_6$$
$$-2(y_4+y_8)]\pm.000,099R_8 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7r).$$

$$A=\frac{x}{20}[(y_0+y_2+y_4+y_6)+5(y_1+y_5)+6y_3]\pm.006,62R_6 \text{ (Weddle's}$$
$$\text{formula)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8r).$$

$$A=\frac{x}{420}[17(y_0+y_{12})+189(y_2+y_{10})+297(y_4+y_8)-256(y_3+y_9)-74y_6]$$
$$\pm.002,55R_8 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9r).^*$$

*Note that ordinates are not all equidistant, but would be so if certain missing ordinates, as indicated by the subscripts, be supplied.

$$A=\frac{x}{5670}[217(y_0+y_8)+1,024(y_1+y_3+y_5+y_7)+352(y_2+y_6)+436y_4] \pm .002,02R_8 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14r).$$

$$A=\frac{x}{840}[41(y_0+y_6)+216(y_1+y_5)+27(y_2+y_4)+272y_3] \pm .008,39R_8 \quad (12r).$$

$$A=\frac{x}{280}[15(y_0+y_6)+64(y_1+y_3+y_5)+29(y_2+y_4)] \pm .006,05R_6 \quad . \quad (18r).$$

$$A=\frac{x}{420}[25(y_0+y_6)+81(y_1+y_2+y_4+y_5)+46y_3] \pm .017,1R_6 \quad . \quad . \quad (19r).$$

$$A=\frac{x}{2100}[49(y_0+y_{12})+288(y_1+y_5+y_7+y_{11})+448(y_3+y_9)-27(y_2+y_{10}) -63(y_4+y_8)+134y_6] \pm .000,071,2R_{10} \quad . \quad . \quad . \quad . \quad . \quad (17r).$$

$$A=\frac{x}{1801800}[41,833(y_0+y_{12})+248,832(y_1+y_5+y_7+y_{11})+395,264(y_3+y_9) +118,416y_6-29,160(y_2+y_{10})-63,909(y_4+y_8)] \pm .000,095,6R_{12} \quad (16r).$$

Formulae expressed in terms of ordinates and the derivatives of the extreme ordinates, wherein $E_n \equiv [f^n(a+x)-f^n(a)]$.

$$A=\frac{x}{30}[7(y_0+y_2)+16y_1]-\frac{x^2}{60}E_2 \pm .115R_6 \quad . \quad . \quad . \quad . \quad . \quad (25s).$$

$$A=\frac{x}{80}[13(y_0+y_3)+27(y_1+y_2)]-\frac{x^2}{120}E_2 \pm .019,1R_6 \quad . \quad . \quad . \quad (26s).$$

$$A=\frac{x}{1890}[217(y_0+y_4)+512(y_1+y_3)+432y_2]-\frac{x^2}{252}E_2 \pm .005,9R_8 \quad . \quad (27s).$$

$$A=\frac{x}{1680}[239(y_0+y_6)+729(y_2+y_4)-256y_3]-\frac{x^2}{168}E_2 \pm .019,6R_8 \quad . \quad (28s).*$$

$$A=\frac{x}{75600}[7,435(y_0+y_{12})+32,768(y_3+y_9)+34,560y_6-19,683(y_4+y_8)] -\frac{x^2}{360}E_2 \pm .002,88R_{10} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29s).*$$

$$A=\frac{x}{13650}[913(y_0+y_{12})+3,402(y_2+y_{10})-2,048(y_3+y_9)+3,888(y_4+y_8) +1,340y_6]-\frac{x^2}{780}E_2 \pm .000,151R_{10} \quad . \quad . \quad . \quad . \quad . \quad . \quad (32s).*$$

$$A=\frac{x}{42000}[3,149(y_0+y_6)+7,776(y_1+y_5)+6,075(y_2+y_4)+8,000y_3] -\frac{x^2}{600}E_2 \pm .000,633R_{10} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35s).$$

$$A=\frac{x}{378378000}[13,929,421(y_0+y_{12})+35,831,808(y_1+y_5+y_7+y_{11}) +25,824,096(y_2+y_{10})+38,174,720(y_3+y_9)+25,511,355(y_4+y_8)$$

*Note that ordinates are not all equidistant, but would be so if certain missing ordinates, as indicated by the subscripts, be supplied.

$$+28,171,584y_6] - \frac{x^2}{2520}E_2 \pm .000,002,8R_{14} \quad . \quad . \quad . \quad . \quad . \quad (39s).$$

$$A = \frac{x}{126}[31(y_0+y_2)+64y_1] - \frac{5x^2}{252}E_2 + \frac{x^4}{15120}E_4 \pm .078,4R_8 \quad . \quad . \quad (42t).$$

$$A = \frac{x}{728}[121(y_0+y_3)+243(y_1+y_2)] - \frac{5x^2}{546}E_2 + \frac{x^4}{65520}E_4 \pm .006,03R_8 \quad (43t).$$

$$A = \frac{x}{32130}[3,937(y_0+y_4)+8,192(y_1+y_3)+7,872y_2] - \frac{x^2}{204}E_2 + \frac{x^4}{257040}E_4$$
$$\pm .001,31R_{10} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44t).$$

$$A = \frac{x}{1635480}[132,761(y_0+y_6)+279,936(y_1+y_5)+264,627(y_2+y_4)$$
$$+280,832y_3] - \frac{25x^2}{11682}E_2 + \frac{x^4}{1401840}E_4 \pm .000,079R_{12} \quad . \quad . \quad (50t).$$

The well known *Simpson's rule* and *Weddle's rule* involve the successive use of *Simpson's formula* and *Weddle's formula* respectively. (See formulae (1r), (2r) and (8r) above). Similar use may, of course, be made of other of the foregoing formulae in y.

Still other formulae may be had in these ordinates by eliminating between two or more equations involving the same set, as between (12r) and (18r). The new formulae thus formed are usually less accurate than those above given, having been formed by considering the remainders zero or negligible, but may be advantageously used in cases where the values of certain ordinates are not readily obtainable.

For ready reference the numerical coefficients in some of the foregoing formulae and their logarithms are given below in decimal form. There may be an error of ± 1 in the last decimal place, as results were obtained by using a 10-place table of logarithms.

Formula (15)

C_8	3.842,785,222	.584,646,111,9
C_6	3.570,612,245	.552,742,690,0
C_4	.866,878,308	9.937,958,135,2—10
C_3	.142,012,987	9.152,328,062,0—10
C_2	.002,962,963	7.471,726,222,9—10
C_1	.000,001,260	4.100,288,905,5—10

Formula (21)

C_{16}	1.450,463,049	.161,506,669,3
C_8	.481,599,060	9.682,685,630,1—10
C_4	.031,604,938	8.499,754,946,5—10
C_2	.000,470,312	6.672,385,673,4—10
C_1	.000,001,383	4.140,906,756,4—10

Formula (22)		
C_{24}	1.686,455,775	.226,974,956,5
C_{12}	.994,333,666	9.997,532,144,1−10
C_8	.384,278,522	9.584,646,111,9−10
C_6	.079,346,939	8.899,530,176,2−10
C_4	.003,095,994	7.490,800,103,7−10
C_3	.000,150,278	6.176,896,253,7−10
C_2	.000,000,592	3.772,322,141,0−10
C_1	.000,000,000	9.185,285,023,3−20
Formula (24)		
C_{18}	1.814,431,370	.258,740,545,7
C_9	.832,473,483	9.920,370,408,9−10
C_6	.018,318,060	8.262,879,471,3−10
C_3	.000,286,220	6.456,699,497,3−10
C_2	.000,010,279	5.011,971,771,6−10
C_1	.000,000,005	1.737,981,960,5−10
Formula (39)		
$C_{12,1}$	1.136,381,334	.055,524,091,1
$C_{6,1}$	.158,693,878	9.200,560,171,9−10
$C_{4,1}$	.024,767,952	8.393,890,090,8−10
$C_{3,1}$	.002,479,592	7.394,380,197,9−10
$C_{2,1}$	.000,024,187	5.383,590,134,1−10
$C_{1,1}$	.000,000,003	1.422,074,122,7−10
Formula (50)		
$C_{6,2}$	1.026,986,573	.011,564,765,5
$C_{3,2}$	.028,081,664	8.448,422,840,3−10
$C_{2,2}$	.001,095,703	7.039,692,767,8−10
$C_{1,2}$	.000,000,611	3.786,354,762,5−10
Formula (51)		
$C_{8,2}$	1.020,853,761	.008,963,532,4
$C_{4,2}$	.020,935,477	8.320,882,870,6−10
$C_{2,2}$	.000,081,779	5.912,642,905,3−10
$C_{1,2}$	.000,000,061	2.784,243,636,6−10
Formula (52)		
$C_{12,2}$	1.033,726,545	.014,405,668,3
$C_{6,2}$	.036,089,574	8.557,381,757,7−10
$C_{4,2}$	.002,503,394	7.398,529,158,3−10
$C_{3,2}$	.000,140,975	6.149,141,792,4−10
$C_{2,2}$	.000,000,611	3.786,169,210,6−10
$C_{1,2}$	.000,000,000	9.222,593,207,8−20

EXAMPLES.

Ex. 1. Construct a five-place table of the values of $\int_0^{n\pi} \frac{\sin x}{x}$ for all integral values of n from 1 to 10, inclusive.

Solution. $\int_0^{10\pi} \frac{\sin x}{x} dx = \sum_{n=0}^{n=10} \int_0^{\pi} \frac{\sin(n\pi+x)}{n\pi+x} dx$. Similarly for $n=9$, 8, 7, etc.

Each value to be found may therefore be derived by the aid of the sum of all the preceding values, already found.

To determine which of the given formulae is sufficiently accurate and at the same time involves the minimum amount of labor is now advisable. Formula (17) has $\pm.0000712R_{10}$ for its remainder term, and, as seems probable, ought to be within the bounds of accuracy required. Expressed more fully this remainder term is

$$\pm \frac{712}{10^7} \cdot \frac{\pi^{10}}{10!} [f^{10}(x_1) - f^{10}(x_2)], \text{ where } 0 \leqq x_1 \leqq \pi \text{ and } 0 \leqq x_2 \leqq \pi.$$

Developing $\sin x/x$ into a power series in x, and differentiating nine times to obtain the tenth derivative, we get another power series in x, the value of which lies somewhere between 0 and $-\frac{1}{8}$ for $0 \leqq x \leqq \pi$. Substituting $-\frac{1}{8}$ for the bracket expression above, we find that the remainder term has a value within the limits of $\pm.000,000,4$. (17) is therefore sufficiently accurate for this value of x, and it can be readily shown that its value for other values of x involved in the problem is less than its value between the limits here considered.

The chordal areas (represented generally by C with a subscript equal to the number of equidistant ordinates employed, less one, the first ordinate being taken at the point $x=0$ and the last at the point $x=X$, where X is the maximum value of x), involved in (17) are,

$$C_{12}=1.850,119, \quad C_6=1.844,651, \quad C_4=1.835,508,$$

$$C_3=1.822,636, \text{ and } C_2=1.785,398.$$

Substituting in (17), $\int_0^{\pi} \frac{\sin x}{x} dx = 1.851,937.$

To check the accuracy of the numerical work done, these values of C may be substituted in (8), and, by doing so, it is found that the same result is obtained by either formula. It is therefore probably safe to employ the simpler formula in computing the remaining terms sought.

It is also instructive to apply some of the other formulae involving

the same values of C, *e. g.*, (9), (10), (11), (13), (16), etc., as well as those involving the second, fourth and sixth derivatives of the function sought, such as (43a), (45a), (48a), (55a), etc.

Ex. 2. Find the common logarithm of 2 from the identity,

$$10\log 2 - \log 1000 = M\int_0^{.024} \frac{dx}{1+x},$$

M being the modulus of the common system of logarithms.

Here the remainder term is easily evaluated, and the selection of the formula to be used is determined largely by the degree of accuracy required. Employing (35s) we get, log2=.301,029,995,663,981,195,213,738,269,3, which we know from the form of the remainder must be correct to not less than 21 decimal places. Expressed in terms of the formula after some simple reductions have been made, the development is,

$$\log 2 = \tfrac{3}{10} + \frac{M}{10}\left(\frac{253}{3,500}\left[\frac{3,149}{64,000} + \frac{7,776}{64,005} + \frac{6,075}{64,008}\right] + \frac{8}{1,771} - \frac{2,277}{10^8.512}\right) \pm .000,633R_{10}.$$

Using (39s) a result correct to thirty places of decimals would have been assured. Should still greater accuracy be desired, the step-by-step process of Ex. 1 may be used.

Ex. 3. Similarly, the logarithm of 2 known, the common logarithm of 3, correct to forty-seven decimal places, may be obtained by (39s) from the identity,

$$8\log 3 + \log 10 - 16\log 2 = M\int_0^{\cdots} \frac{dx}{1+x}.$$

Ex. 4. Obtain the value of $\int_0^{\frac{1}{3}\pi} \frac{xdx}{\sin x(1+\frac{4}{25}\cos^2 x)^{\frac{1}{2}}}$ correct to six decimal places.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

364. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

The English physicist, Hooke, published the discovery contained in the Latin sentence, "Ut tensio sic vis" by the cypher *ceiiinosssttuv*. Preserving the lexicographical order, find which permutation, taking all letters, the Latin sentence is from the cypher.

Solution by H. C. FEEMSTER, A. M., York College, York, Nebraska.

Beginning at the first, and counting all permutations, although letters may be alike, the permutation required is:

$$12\underline{|13}+10\underline{|12}+\underline{|10}+4\underline{|9}+5\underline{|8}+\underline{|7}+3\underline{|6}+3\underline{|5}+\underline{|4}+2\underline{|2}+\underline{|1}$$
$$=79,\ 519,\ 555,\ 109.$$

370. Proposed by E. B. ESCOTT, University of Michigan, Ann Arbor, Mich.

Prove that, if the fraction m/p (p prime) gives a recurring decimal with an even number of digits in the cycle, the sum of the two halves will be composed of 9's. (The special case where the number $=p-1$ was proposed in the MONTHLY, Vol. IV, and answered in Vol. V, p. 11. The proof there given, however, is not complete.) The above property is true of other fractions, *e. g.*, $\frac{1}{77}=.\dot{0}12987\dot{}$, $\frac{1}{133}=\frac{1}{7.19}=.\dot{0}07518796,992481203\dot{}$. Find for what fractions this is true.

I. Solution by F. H. SAFFORD, Ph. D., University of Pennsylvania.

1°. To prove that the sum of the two halves will be composed of 9's. Let the number of digits in the cycle be $2n$, then the number of 9's in the denominator of the unreduced common fraction will be $2n$, and this sequence is divisible by the sequence of n 9's, *e. g.*, 999999 is divisible by 999. It follows that the unreduced numerator must be also divisible by the sequence of n 9's. Suppose the original cycle is $\dot{a}bcde\dot{f}$, the fraction may be written

$$\frac{(abc)\,(1000)+(def)}{999999},\ \text{or}\ \frac{(abc)\,(999)+(abc)+(def)}{999999},$$

in which (abc), etc., represents a number of three digits and not a product. The part $(abc)+(def)$ must be divisible by 999 as shown by the preceding.

Not all of the digits a, b, c, d, e, f, can be 9's, so that the sum is less than twice 999 and is thus 999. Hence for a given (abc) the corresponding (def) is fixed and has the specified property. The proof is extensible to any case and does not require that the recurring decimal be perfect.

2°. The results of the indicated cancellations are:

$$\frac{a+1}{11}, \quad \frac{(ab)+1}{101}, \quad \frac{(abc)+1}{1001}, \text{ etc.}$$

Of such fractions, those denominators (except the first) having an even number of digits are composite, also others such as $10001=73.137$. The numerators $(abc\ldots)+1$ are unrestricted as might be anticipated, so that fractions having denominators of the form 10^n+1 have the required property. The illustration $\frac{1}{77}=.\dot{0}12987$ came from $\frac{012+1}{1001}$; the fraction $\frac{1}{133}$ originally had the numerator $007518796+1$ and the denominator 10^9+1.

II. Solution by the PROPOSER.

Let the number of digits in the cycle be $2n$. Let d be the value of the recurring decimal.

$d=.abc\ldots(n \text{ digits})\ldots{}^{\alpha\beta\gamma}\ldots(n \text{ digits})\ldots$

Then $10^{2n}d=abc\ldots{}^{\alpha\beta\gamma}\ldots\ .abc\ldots{}^{\alpha\beta\gamma}\ldots$; $d=.abc\ldots{}^{\alpha\beta\gamma}\ldots$

Subtracting, we have, $(10^{2n}-1)d=abc\ldots{}^{\alpha\beta\gamma}\ldots$ an integer.

$$\therefore d=\frac{abc\ldots{}^{\alpha\beta\gamma}\ldots}{10^{2n}-1}=\frac{m}{p} \text{ by hypothesis.}$$

Then $\frac{m(10^{2n}-1)}{p}=\frac{m(10^n-1)(10^n+1)}{p}$ is an integer $=abc\ldots{}^{\alpha\beta\gamma}\ldots$

p cannot divide 10^n-1 since in that case the period of m/p would be n and not $2n$. Therefore p is a factor of 10^n+1.

$$\begin{aligned} 10^n(m/p) &= abc\ldots\ .{}^{\alpha\beta\gamma}\ldots abc\ldots \\ (m/p) &= \qquad .abc\ldots{}^{\alpha\beta\gamma}\ldots \\ \hline (10^n+1)(m/p) &= abc\ldots \ = \text{an integer by proof just given.} \end{aligned}$$

Therefore, the sum of the two decimals $.{}^{\alpha\beta\gamma}\ldots$ and $.abc\ldots$ must be $.999\ldots$

The answer to the second part of the question follows at once from the above. It is sufficient if p instead of being a prime is *any* factor of 10^n+1. $10^3+1=7.11.13$. Therefore if $p=77$, 91, 143, or 1001, a/p will have six figures in the cycle and the two halves will be complements.

III. Solution by C. A. LAISANT, Paris, France.

If the fraction m/pp' gives a periodical decimal fraction with an even number $2n$ of digits, and if we write the period $A.10^n+B$, we have

$$\frac{m}{pp'}=\frac{A.10^n+B}{(10^n-1)(10^n+1)}.$$

10^n-1 cannot be mult. p and mult. p', because the period would then have n digits, instead of $2n$.

Suppose 10^n+1 mult. p and mult. p', hence mult. pp' (supposing p, p' primes together). Writing $10^n+2=qpp'$, we have

$$mq=\frac{A.10^n+B}{10^n-1}=A+\frac{A+B}{10^n-1}.$$

But $A<10^n$, $B<10^n$. Then, $A+B=10^n-1$, and the property is true.

If 10^n+1=mult. p, and if p', divisor of 10^n-1, is 3 or 9, the sum $A+B$ is a number with similar digits.

Ex. $\frac{1}{21}=\frac{1}{7.3}=0,047619...$ $\quad 047+619=666.$

$\frac{1}{63}=\frac{1}{7.9}=0,015373...$ $\quad 015+373=888.$

Also solved by J. Scheffer.

H. C. Feemster should have received credit for solving 368.

GEOMETRY.

390. Proposed by PROF. R. C. ARCHIBALD, Brown University, Providence, R. I.

Find, geometrically, and without introducing focal properties, the locus of the vertices of the conjugate parallelograms of an ellipse.

Solution by the PROPOSER.

The solution which I had in mind when I proposed this problem was the following: By the elementary ideas of a *stretch* in projective geometry, if the radius of a circle, a, be b, and the stretch ratio be $\frac{a}{b}(a>b$, say), a will be transformed into an ellipse with semi-axes of lengths a and b. Corresponding to any conjugate parallelogram of the ellipse will be a square circumscribing a; and since the locus of the vertices of all such squares is another circle concentric with a and of radius $\sqrt{(2)}b$, the stretch which transforms a into the ellipse with axes a and b will transform the concentric circle with radius $\sqrt{(2)}b$ into the ellipse with axes $\sqrt{(2)}a$, $\sqrt{(2)}b$. And this is the required locus, as found *analytically* by Mr. Scheffer (AMERICAN MATHEMATICAL MONTHLY, Vol. XIX, p. 54, March, 1912).

398. Proposed by C. N. SCHMALL, New York City.

In a square $ABCD$ draw the diagonal AC. Now bisect AD in G and draw GB cutting AC in H. Prove that $\triangle AGH=\frac{1}{2}\triangle CGH=\frac{1}{3}\triangle ABG=\frac{1}{4}\triangle BCH$.

Solution by A. H. HOLMES, Brunswick, Maine, and G. I. HOPKINS, A. M., Manchester, New Hampshire.

Since $BC=2AG$ by construction, and $\angle CBH=\angle GAH$, $CH=2AH$. But the altitudes of triangles AGH and CGH are the same.

$\therefore \triangle AGH=\frac{1}{2}\triangle CGH$.

For the same reason as above, $BH=2GH$.

$\therefore BG=3GH$.

The altitudes of triangles AGH and ABG being the same, $\triangle AGH=\frac{1}{3}\triangle ABG$. Since the homologous sides in triaugles AGH and BCH are one-half the size in the former of those in the latter, $\triangle AGH=\frac{1}{4}\triangle BCH$.

Also solved by J. Scheffer, Francis Rust, M. A. Muzzy, and H. C. Feemster.
A. M. Harding should have received credit for solving 397.

399. Proposed by J. K. ELLWOOD, Superintendent of Schools, Lucas, Kansas.

A race track is to be composed of two tangents and the arc of the circle which is concave towards the point of intersection of the two tangents, each tangent and the arc of the circle being 1 mile. What is the radius of the circle?

Solution by CHRISTIAN HORNUNG, Heidelberg University, Tiffin, Ohio.

Let AT and BT be the tangents, and ACB the arc composing the track; O the center, and OA the radius of the circle.

Let $AOT=\theta$, and $AO=r$. Then arc $ACB=(2\pi-2\theta)r=1$, and $\tan\theta=1/r$; whence $\tan\theta=2(\pi-\theta)$.

This equation solved by approximation gives

$$\theta=74^\circ\ 46'.18. \qquad r=\frac{1\text{ mile}}{\tan\theta}=\frac{5280\text{ feet}}{\tan\theta}=1437.45\text{ feet}.$$

Also solved by J. E. Sanders, A. M. Harding, J. Scheffer, H. Prime, A. H. Holmes, Elmer Schuyler, E. B. Escott, and H. C. Feemster.

CALCULUS.

321. Proposed by ARTEMAS MARTIN, Ph. D., LL. D., United States Coast and Geodetic Survey Office, Washington, D. C.

To a person in a boat at the center of a circular pond the bottom appears to be perfectly level. What is the actual form of the bottom of the pond, the depth of the water at the center being a feet, and the distance of the eyes of the observer from the surface of the water being b feet. [From the *Mathematical Visitor*, Vol. 2, No. 2, p. 62.]

Solution by E. B. ESCOTT, Ann Arbor, Michigan.

Let A be the bottom of the pond at the center, OX the surface of the water, B the observer. Then $\sin\theta=m\sin\phi$.

Let $OQ=q$, $\sin\theta=\dfrac{q}{\sqrt{(q^2+b^2)}}$, $\sin\phi=\dfrac{q}{m\sqrt{(q^2+b^2)}}$;

$$QP=QR=\frac{a}{b}.\ BQ=\frac{a}{b}\sqrt{(q^2+b^2)};\ x=q+\frac{aq}{bm};\ y=QP\cos\phi$$

$$=a\sqrt{1+\frac{(m^2-1)x^2}{(a+bm)^2}};\ i.\ e.,\ \frac{y^2}{a^2}-\frac{x^2}{\dfrac{(a+bm)^2}{m^2-1}}=1,$$

hyperbola, center at O, AB principal axis.

An excellent solution was also received from H. C. Feemster.
This problem properly belongs to the Miscellaneous Department.

322. Proposed by E. B. ESCOTT, University of Michigan.

Find the equation of the curve such that the solid of revolution generated by revolving it about the x-axis shall have a volume equal to the m/nth part of the volume of the circumscribed cylinder.

Solution by H. PRIME, Boston, Massachusetts, and J. SCHEFFER, A. M., Hagerstown, Maryland.

Let $\pi\int y^2dx=\pi\ (m/n)y^2x$.

Differentiating and arranging, $2m\ dy/y=(n-m)\ dx/x$.

Hence by integrating, $2m\log y=(n-m)\log x+\log c$.

$\log y^{2m}=\log x^{n-m}+\log c$, $y^{2m}=a^m x^{n-m}$ and $y^2=ax^{n/m-1}$.

This is the equation of the family of parabolas, including the case when $m/n=1/3$ and $y^2=ax^2$. Thus we have the familiar relation of the cone and its circumscribing cylinder.

Also solved similarly by M. A. Muzzy, and Elijah Swift.

323. Proposed by C. N. SCHMALL, New York City.

From what height must an elastic ball be dropped in order that, after impact with the hard surface of the sidewalk, it may rebound to a given altitude a in the least possible time from the moment of descent?

Solution by J. SCHEFFER, A. M., Hagerstown, Maryland.

Let x be the height. Then will the velocity at the moment the ball strikes the surface$=\sqrt{(2gx)}$, and the time, t_1, $=\sqrt{(2x/g)}$. It rebounds now with a velocity$=e\sqrt{(2gx)}$, e being the coefficient of elasticity. Denoting the time it reaches the height a by t_2, we have $a=e\sqrt{(2gx)}.t_2-(gt_2^2/2)$, whence $t_2=\sqrt{(2/g)}]e\sqrt{x}\pm\sqrt{(e^2x-a)}]$. Consequently the total time is

$$T=t_1+t_2=\sqrt{\frac{2x}{g}}+\sqrt{\frac{2}{g}}[e\sqrt{x}\pm\sqrt{(e^2x-a)}].$$

Putting $dT/dx=0$, we get $x=\frac{a(1+e)^2}{e^2(1+2e)}$.

Also solved by C. N. Schmall, and H. Prime. In their solutions, these gentlemen use $e=1$.

324. Proposed by V. M. SPUNAR, Chicago, Ill.

A parabola slides between two rectangular axes; find (a) the locus of the focus, and (b) the locus of the vertex.

I. Solution by ELIJAH SWIFT, Princeton University.

Take the parabola in the form $y^2=4ax$. The equation of any two perperdicular tangents may be written $y=mx+(a/m)$ and $y=-(1/m)x-am$. The distances of these from the focus are $Y=(a/m)\sqrt{(1+m^2)}$ and $X=a\sqrt{(1+m^2)}$, and these quantities would therefore be the coördinates of the focus of a moving parabola, if the two perpendicular tangents were axes. Eliminating m, we get as the locus

$$x^2y^2=a^2(x^2+y^2), \text{ or in polar coördinates, } r=\frac{2a}{\sin 2\phi}.$$

In case (b), $Y=\frac{a}{m\sqrt{(1+m^2)}}$, $X=\frac{am^2}{\sqrt{(1+m^2)}}$.
Elimination of m gives us $x^2y^2(x^3+y^3)^3=a^6$.

261. Proposed by V. M. SPUNAR, M. and E. E., Chicago, Ill.

A man six feet high, walking at a rate of 100 yards a minute, crosses a muddy road close behind a wheel of a carriage which is going thrice as fast and in a direction at right angles to that of the man's motion. The diameter of the wheel is five feet. If, when the man is four feet from the middle of the wheel the mud is splashed up to the hight of seven feet, will any of it touch him? Unsolved in *Educational Times*.

Solution by H. PRIME, Boston, Massachusetts.

Let ϕ be the angle of elevation, v the velocity, of particles of mud leaving the ground at A; let the line ABH be the path of the wheel, $DEFGHI$ the man's path, the man being at D when the wheel touches A, at F when it touches B, and the mud is 7 feet high. Then $BF=4$ feet, $BH=3FH-\frac{5}{2}$, $16-FH^2=(3FH-\frac{5}{2})^2$, $FH=1.989960$, $BH=3.469879$.

The man is at three critical points in his path,—at E when the mud is 6 feet high rising, at G when it is 6 feet high falling, at I when the mud is striking the ground. While the man is between E and G, no mud can touch him, for it is more than 6 feet high. Between D and E only rising mud can hit him, between G and I only falling mud.

If $g=32$, the mud rises 7 feet, or falls 7 feet, in $\sqrt{\frac{7}{16}}$ seconds, and rises the last foot, or falls the first foot, in $\frac{1}{4}$ second. While the mud is rising, the carriage goes from A to B, 15 feet a second. Hence $AB=15\sqrt{\frac{7}{16}}=9.921568$, $AH=13.391447$; also $EF=FG=1.25$, $DF=FI=3.307189$; $EH=3.239560$, $GH=0.739560$, $HI=1.317630$; hence $AE=13.778007$, $AG=13.412149$, $AI=13.456409$.

For determining the limiting values of ϕ or v we have the equation of the parabola in which a projected particle of mud moves, A being the origin, $y=x\tan\phi-gx^2/2v^2\cos^2\phi$; also the maximum height$=v^2\sin^2\phi/2g=7$. Whence $\tan\phi=[14\pm\sqrt{(196-28y)}]/x$.

At E, $x=AE$, $y=6$. Hence $\phi=32^\circ\ 17'\ 43''$ or $v=39.6158$ feet a second.

At G, $x=AG$, $y=6$. Hence $\phi=55^\circ\ 11'\ 30''$ or $v=25.7787$ feet a second.

At I, $x=AI$, $y=0$. Hence $\phi=64^\circ\ 19'\ 54''$ or $v=23.5149$ feet a second.

Thus, if $32^\circ\ 17'\ 43''<\phi<55^\circ\ 11'\ 30''$, or if $\phi>64^\circ\ 19'\ 54''$, none of the mud touches the man. For in the former case it passes over his head, in the latter it does not reach to him. The man is hit between D and E if $\phi<32^\circ\ 17'\ 43''$, or between G and I if $55^\circ\ 11'\ 30''<\phi<64^\circ\ 19'\ 54''$.

Also solved by the Proposer.

262. Proposed by V. M. SPUNAR, M. and E. E., Chicago, Ill.

A hemispherical shell, whose radius is equal to the mean radius of the earth and whose thickness is one centimeter, is constructed of a matter whose density is equal to the mean density of the earth. A particle starts from rest at the center of the shell under the action of the attraction of the shell. Express as the decimal of a year the time it takes the particle to reach the surface of the shell, and find the velocity in centimeters per second of the particle just before it reaches the shell. Unsolved in *Educational Times*.

I. Solution by the PROPOSER.

$$\frac{\text{The total volume of the earth}}{\text{The volume of the shell}}=\tfrac{2}{3}R, \text{ nearly; } R=\frac{40{,}000{,}000}{2\pi}m=\frac{2}{\pi}10^9 c\,m,$$

the mean radius of the earth.

$$\therefore \frac{\text{The mass of the earth, } M'}{\text{The mass of the shell, } M}=\frac{147\times10^9 R^2}{M}=\tfrac{2}{3}R,$$

at the same mean density of the earth.

$$\therefore M=\tfrac{1}{3}\times147\times10^9 Rkg\ (R \text{ in meters})=\frac{441\times10^{19}}{\pi}g.$$

The center of gravity, O, of the mass M is the center of gravity of the circular arc on the axis of symmetry (X-axis) at the distance $(2/\pi)R$ from O', the center of the shell.

The force of attraction, F, exerted by O on the particle (mass=1) in any position P between $O'O$ is CM/x^2, where x=abscissa of P and C, the constant of gravitation$=6.5\times10^{-8}$ (Baily). C becomes, however, 1 if cm.—sec. would be chosen and $1.537\times10^7 g$ as unity of the mass, M. Hence, $MC=\frac{441\times10^{15}}{1537\pi}$. Then,

$$\int_{v_0}^{v_1} v\,dv = \int_{x}^{2R/\pi} F\,dx = MC\int_{x}^{2R/\pi} \frac{dx}{x^2}. \quad \therefore \tfrac{1}{2}(v_1^2 - v_0^2) = \frac{MC(2R-\pi x)}{2xR}.$$

$$\therefore v_1^2 = \frac{MC(2R-\pi x)}{Rx} + v_0^2 = \left(\frac{dx}{dt}\right)^2,$$

where the initial velocity $v_0 = 0$.

It is evident, by the last formula, that the motion of the particle is such that if it reaches the point $x_1 = 2R/\pi$ $(dx/dt = 0)$ will turn there and move in the other direction with its maximum velocity at O (the latter not necessarily $= \infty$ as the formula shows, which is only true for $0 < x \leq x_1$).

$$\therefore t_1 = \sqrt{\frac{R}{MC}}\int_0^{2R/\pi} \frac{\sqrt{[(x)\,dx]}}{\sqrt{[2R-\pi x]}} = \sqrt{\left(\frac{R^3}{MC\,\pi}\right)} = \sqrt{\left(\frac{8\times 1537\times 10^{12}}{441\,\pi^3}\right)}$$

$= 948{,}283$ seconds $= 10$ days, 23 hours, 24 minutes, 42 seconds, is the time the particle takes to reach the center of gravity; and that, the particle takes to move from O to the shell itself,

$$t_2 = \sqrt{\left[\left(\frac{R}{\pi}\right)^3 \frac{1}{MC}\right]}\{\cos^{-1}(\pi-3) - \sqrt{[(\pi-2)(4-\pi)]}\}$$

$= 259{,}916$ seconds $= 3$ days, 0 hours, 11 minutes, 56 seconds.

Therefore, the total time in question, $T = t_1 + t_2 = 1{,}208{,}199$ seconds $=$ 13 days, 23 hours, 36 minutes, 38 seconds $= \dfrac{1{,}208{,}199}{31{,}556{,}936} = .038286+$ year.

From symmetry of the motion we infer that the velocity, v_2, of the particle just before reaching the shell is the same (with respect to its scalar value) as that at the point $x_2 = R(\pi-2)/\pi$.

$$\therefore v_2 = \sqrt{\left(\frac{\pi\,MC(4-\pi)}{R(\pi-2)}\right)} = 1000\sqrt{\left(\frac{441(4-\pi)\pi}{1537(\pi-2)2}\right)} = 582.15 \text{ cm/sec.}$$

II. Solution by S. G. BARTON, Ph. D., Clarkson School of Technology, Potsdam, New York.

The component of the attraction along the axis of symmetry at any point on this axis within the shell is

$$X = \frac{\pi\rho r}{c^2}\left[\frac{r^2 - c^2 + y^2}{y}\right]_{y_0}^{y_1}$$

[Pierce, *Newtonian Potential Functions*, page 12], where ρ = mass of 1cc. of

matter, r is the radius of the sphere, c is the distance of the point from the center, and y is the distance from the point to the element of area. The limits are $y_0=\sqrt{[r^2+c^2]}$, $y_1=r-c$.

$$X=\frac{\pi\rho r}{c^2}\left[\frac{r^2-c^2+(r-c)^2}{r-c}-\frac{r^2-c^2+r^2+c^2}{\sqrt{[r^2+c^2]}}\right]=\frac{2\pi\rho r^2}{c^2}\left[1-\frac{r}{\sqrt{[r^2+c^2]}}\right].$$

$$X=\frac{d^2c}{dt^2}.\quad 2\frac{dc}{dt}\frac{d^2c}{dt^2}=4\pi\rho r^2\left[\frac{1}{c^2}-\frac{r}{c^2\sqrt{[r^2+c^2]}}\right]\frac{dc}{dt}.$$

$$\left(\frac{dc}{dt}\right)^2=v^2=4\pi\rho r^2\left[-\frac{1}{c}+\frac{\sqrt{[r^2+c^2]}}{rc}\right]+k_1.$$

$v=0$ when $c=0$. $-\frac{1}{c}+\frac{\sqrt{[r^2+c^2]}}{rc}$ is indeterminate for $c=0$. Its value is 0. Therefore $k_1=0$.

$$v=2r\sqrt{[\pi\rho]}\left[\frac{\sqrt{[r^2+c^2]}-r}{rc}\right]^{\frac{1}{2}}$$

where $c=r$. $v=2\sqrt{[\pi\rho r]}\sqrt{[\sqrt{(2)}-1]}$. But the attraction of the earth for a point at its surface is $\frac{4}{3}\pi\rho r=980$cm. per second.

$\therefore \pi\rho r=735$. $\therefore v=34.9$ cm. per second.

$v=\frac{dc}{dt}$. $\therefore \int\left[\frac{c}{\sqrt{[r^2+c^2]}-r}\right]^{\frac{1}{2}}dc=2\sqrt{[\pi\rho r]}t+k_2$, which, when integrated, gives t. $t=0$ when $c=0$.

PROBLEMS FOR SOLUTION.

ALGEBRA.

377. Proposed by E. B. ESCOTT, Ann Arbor, Michigan.

Expand in series $(1+x)^{1/x}$.

378. Proposed by C. N. SCHMALL, New York City.

Given $\sin^{-1}\frac{x}{a}+\sin^{-1}\frac{x}{b}+\sin^{-1}\frac{x}{c}=\pi$, solve for x.

379. Proposed by C. N. SCHMALL, New York City.

Given $y=ax+b$. If the values of x vary in an arithmetical progression, show that the value of y varies likewise.

GEOMETRY.

406. Proposed by DR. R. K. MORLEY, University of Illinois.

Given the lengths of a pair of conjugate diameters of an ellipse and the angle between them; to construct (with ruler and compass) the axes of the ellipse, *i. e.*, find their lengths and the angles they make with the given diameters.

407. Proposed by S. LEFSCHETZ, Ph. D., University of Nebraska.

To construct a right triangle knowing the sum of the sides of the right angle, and the sum of one of them plus the hypotenuse.

408. Proposed by ELMER SCHUYLER, Brooklyn, New York.

Given a point A on a circle and a chord of the circle; to draw a chord through A so that it shall be bisected by the given chord.

CALCULUS.

329. Proposed by C. N. SCHMALL, New York City.

Show that the general linear differential equation $\frac{d^ny}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+P_2\frac{d^{n-2}y}{dx^{n-2}}+\ldots+P_{n-1}\frac{dy}{dx}+P_ny=V$, where P_1, P_2, P_3, ..., P_{n-1}, P_n, V are known functions of x (Edwards, *Integral Calculus*, p. 243), has a solution in the form $y=v_1\int v_2dx\int v_3dx\ldots\int\frac{Vdx}{v_1v_2v_3\ldots v_{n-1}v_n}$.

330. Proposed by C. N. SCHMALL, New York City.

X is a homogeneous function, in the nth degree, of x, y, z; Y is any function of u, v, w. If $ux=vy=wz=1+X\ldots(1)$, prove that (by Euler's Theorem), $x\frac{\partial Y}{\partial x}+y\frac{\partial Y}{\partial y}+z\frac{\partial Y}{\partial z}+\frac{n}{x}\frac{\partial Y}{\partial u}+\frac{n}{y}\frac{\partial Y}{\partial v}+\frac{n}{z}\frac{\partial Y}{\partial w}$

$$=(n-1)\left(u\frac{\partial Y}{\partial u}+v\frac{\partial Y}{\partial v}+w\frac{\partial Y}{\partial w}\right).$$

NOTES AND NEWS.

Editor Slaught attended the International Congress of Mathematicians at Cambridge, England. F.

We learn from *School Mathematics and Science* that Professor J. L. Gilpatrick of Denison University, died recently. Professor Gilpatrick was a subscriber of the MONTHLY from its beginning. F.

Dr. G. E. Wahlen, who has been editing the Department of Number Theory and Diophantine Analysis, will go to Europe on a year's leave of absence. Material for his department should, therefore, be sent to Editor Finkel.

The first part of volume 5, tome II, of the French mathematical encyclopedia appeared recently. This part contains 160 pages and is devoted to functional operations and functional equations, and to trigonometic interpolation. The last seven pages are devoted to the beginning of the article on spherical functions. According to a recent announcement in the *Jahresbericht der Deutschen Mathematiker-Vereinigung*, the first part of volume 2, tome IV, has also appeared. This part contains three articles entitled, respectively, Geometric foundations of statics, Geometry of masses, and Cinematics. A very large number of additional parts are announced as in press. Although more than twenty parts have been issued no volume has yet been completed. M.

Henri Poincaré, the most eminent French mathematician and physicist, died on July 17, 1912. He was born at Nancy, France, April 29, 1854, and he rose to fame as a young man in a meteoric manner. For a number of years he has been generally acknowledged as the most eminent mathematician, and numerous biographical sketches have appeared during recent years. The most complete extant sketch of his life work is probably the one published in 1909 by Ernest Lelon under the title "Biographie bibliographie analytique des ecrits," which contains about 80 pages, and was issued by the noted firm of Gauthier-Villars of Paris, France. In January of 1909, Poincaré was received as a member of the forty immortals, and, on this occasion, the director of the Académie Française, M. Masson, gave a comprehensive sketch of his life. A translation of this appeared in the September, 1909, number of *Popular Science Monthly*. A more recent sketch prepared by E. E. Slosson appeared in the October 5, 1911, number of *The Independent* under the title "Twelve major prophets of to-day — III." M.

At the meeting of the National Education Association in Chicago, the round table conference of the Secondary Section was held on Wednesday, July 10, at 9 o'clock. The chairman is Charles M. Austin of the Central High School, Minneapolis. The amended report of the Committee of Fifteen on Geometry Syllabus was presented by Professor E. R. Hedrick of the University of Missouri, who is chairman of the sub-committee on the syllabus proper. Since the report was presented in San Francisco last July full and careful consideration has been given to all criticisms and suggestions and in the light of these numerous modifications and additions have been made, and it is in this completed form that the report will now be given by Professor Hedrick. This revised form of the report has been printed and is in process of distribution through the Department of Education in Washington. Word has just been received that the edition of 5000 copies is nearly exhausted, which would indicate that another edition will need to be issued after the Chicago report has been made. The historical introduction was included in the proceedings of the San Francisco meeting. S.

ERRATA.

Page 27, line 5, Lefschetz's article, for "Caily" read Cayley.
Page 34, counting from top of page, omit line 6.
Page 59, last line, for + read −.
Page 82, problem 376, for "$a_n - k$" read a_{n-k}; and in last line, for "a_n" read a_8.
Page 102, third paragraph, third line, for "$\angle BOP$" read $\angle AOP$.
Same paragraph, fifth line, for "P'_1P_1B" read P'_1P_1A.
Page 105, fifth line, for "P'_1P_1B" read P'_1P_1A.

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A LABOR-SAVING DEVICE FOR SERIAL MULTIPLICATION OR DIVISION BY MEANS OF AN ARITHMOMETER IN CASES WITH SMALL DIFFERENCES OF CONSECUTIVE RESULTS.*

By E. R. MARSHALL, Ph. D., New York City.

INTRODUCTION.

In computing the present value of future expense charges on annual dividend policies for a life insurance company in New York, I had to multiply a constant for every five years of age, representing a certain percentage of the mean of the office premium and the net premium for ordinary life, by a decreasing annuity for the different policy ages. The results obtained were taken only to the nearest integer. As there happened to be repetitions of some integers and omissions of others with what seemed to be absolute irregularity, I thought of trying to determine the limits between the factors giving one integer and those giving the next. This led me to devise a method of serial multiplication by means of the arithmometer (later extended also to serial division) for such and similar kinds of work, which eliminates most of the labor required in the ordinary methods if the differences of the consecutive results do not exceed a few units,— and a large proportion of it for differences greater than a few units. The present paper is an explanation of this device, which, I believe, may prove of interest also to other actuarial computers, as well as to mathematicians in general.

SERIAL MULTIPLICATION.

I shall first treat of serial multiplication when used for finding the nearest or the highest integers of a series of consecutive products with a constant factor, whose differences do not exceed a few units — the actual case that gave rise to the discovery of the device.

*Presented to the American Mathematical Society at its April meeting.